STATISTIK INDUSTRI 1

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Distribusi Peluang Kontinyu

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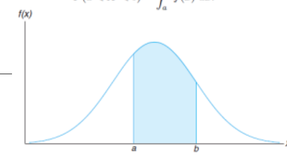
$$P(a < X < b) = \int_a^b f(x)\, dx.$$



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Distribusi Peluang Kontinyu

- Rata-rata dan Variansi
  - Rumus Umum:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

Distribusi Peluang Diskrit dan Kontinyu

UNIFORM

Distribusi Diskrit Uniform

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, $x_1, x_2, \ldots, x_n$, has equal probability. Then,

$$f(x_i) = 1/n$$

$$\mu = E(X) = \frac{b+a}{2}$$

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Distribution	Random Variable X	Possible Values of X	Distribution Function Fx(a) = P(X=a)	Mean E(X)
Uniform	Realization of $x_1, x_2, \ldots, x_n$	$x_1, x_2, \ldots, x_n$	$1/n$	$\frac{b+a}{2}$

Distribusi Diskrit Uniform

- Contoh:
  - Suatu bacth produk terdiri dari nomor serial, dengan nomor urut pertama terdiri dari 0 sampai dengan 9. Jika salah satu produk diambil secara acak, maka X adalah munculnya nomor serial dengan angka pertama tersebut masing-masing nomor (R={0,1,2,...,9) memiliki peluang 0,1.

$$f(x) = \frac{1}{10} = 0{,}1$$

$$\mu = \frac{(9+0)}{2} = 4{,}5$$

$$\sigma^2 = \frac{(9-0+1)^2 - 1}{12} = 8{,}25$$

Distribusi Kontinyu Uniform

The density function of the continuous uniform random variable X on the interval $[A, B]$ is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\mu = \frac{A+B}{2} \text{ and } \sigma^2 = \frac{(B-A)^2}{12}$$

Distribusi Kontinyu Uniform

- Contoh:
  - Variabel acak kontinyu menotasikan pengukuran arus pada kawat tembaga dalam miliamper. Jika diketahui bahwa f(x)=0,05 untuk $0 \leq x \leq 20$. Berapakah peluang pengukuran arus berada antara 5 dan 10 mA.
  - $P(5 < X < 10) = \int_5^{10} f(x)\,dx = 5(0{,}05) = 0{,}25$
  - *Rata-rata dan Variansi distribusi uniform arus kawat tembaga: a=0, b=20*
    - $\mu = E(X) = \frac{(0+20)}{2} = 10mA$
    - $\sigma^2 = V(X) = \frac{(20-0)^2}{12} = 33{,}33\ mA$
    - $\sigma = 5{,}77\ mA$

Distribusi Peluang Kontinyu

GAMMA

Distribusi Gamma

- Diaplikasikan pada masalah antrian dan masalah keandalan (reliabilitas).
- *Time / space occuring until a specified number of Poisson events occur*
- Fungsi gamma: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx, \quad \text{for } \alpha > 0$
- Properti fungsi gamma:

(a) $\Gamma(n) = (n-1)(n-2)\cdots(1)\Gamma(1)$, for a positive integer n

(b) $\Gamma(n) = (n-1)!$ for a positive integer n.

(c) $\Gamma(1) = 1$.

(d) $\Gamma(1/2) = \sqrt{\pi}$.

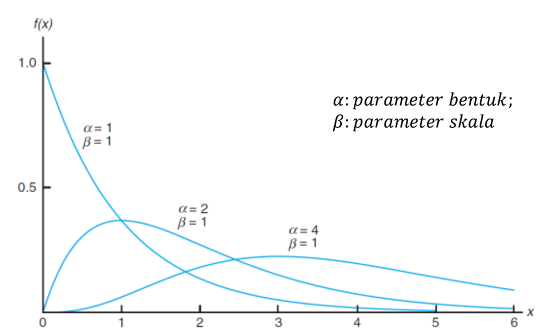
Distribusi Gamma

- Fungsi distribusi gamma:

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$

- β: *waktu rata − rata antar kejadian*
- α: *jumlah kejadian yang terjadi berturutan pada waktu/ruang tertentu*
- λ: *jumlah kejadian per unit waktu/ruang* ($\lambda = 1/\beta$)
- x: *nilai random variabel* (lama waktu atau luasan area hingga kejadian berikutnya)



α: *parameter bentuk*;
β: *parameter skala*

Distribusi Gamma

- Rata-rata dan Variansi:

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2$$

- Jika X_1 dan X_2 adalah variabel acak yang independen, dan $X_1 \sim Gamma\ (\alpha_1, \beta)$; $X_2 \sim Gamma\ (\alpha_2, \beta)$, maka $X_1 + X_2 \sim Gamma\ (\alpha_1 + \alpha_2, \beta)$
- Sehingga, jika $X_i \sim Gamma\ (\alpha_i, \beta), for\ i = 1, \dots, k$, maka $(X_1 + \cdots + X_k) \sim Gamma\ (\alpha_1 + \cdots + \alpha_k, \beta)$

Distribusi Peluang Kontinyu

EKSPONENSIAL

Distribusi Eksponensial

- Bentuk khusus dari distribusi peluang gamma ($\alpha = 1$)
- *Time to arrival or time to first poisson event problems*
- Diaplikasikan pada permasalahan waktu antar kedatangan pada fasilitas jasa, life time / waktu kegagalan komponen, *survival time,* dan waktu respon komputer

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\beta > 0$

$$\mu = \beta \text{ and } \sigma^2 = \beta^2$$

Distribusi Eksponensial

- Eksponensial menganut proses Poisson (λ: laju kedatangan)
- $X \sim Exp(\lambda)$:

$$P(X \geq a) = \int_a^\infty \lambda e^{-\lambda x}\, dx = e^{-\lambda a}$$

$$\mu = \frac{1}{\lambda};\ \sigma^2 = 1/\lambda^2$$

  – $\lambda = 1/\beta$

- Karakter penting: ***memoryless property***
  – Pada permasalahan life time (hingga terjadi break down / failure / kerusakan), misal life time dari lampu, TV, kulkas
- Kerusakan yang diakibatkan oleh pemakaian berkala (misal pemakaian mesin), tidak berlaku distirbusi eksponensial. Lebih tepat menggunakan distribusi GAMMA atau distribusi WEIBULL

Contoh: Gamma

Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard? The Poisson process applies, with time until 2 Poisson events following a gamma distribution with $\beta = 1/5$ and $\alpha = 2$. Denote by X the time in minutes that transpires before 2 calls come. The required probability is given by

$$P(X \leq 1) = \int_0^1 \frac{1}{\beta^2} x e^{-x/\beta}\, dx = 25 \int_0^1 x e^{-5x}\, dx = 1 - e^{-5}(1 + 5) = 0.96.$$

Contoh: Gamma

In a biomedical study with rats, a dose-response investigation is used to determine the effect of the dose of a toxicant on their survival time. The toxicant is one that is frequently discharged into the atmosphere from jet fuel. For a certain dose of the toxicant, the study determines that the survival time, in weeks, has a gamma distribution with $\alpha = 5$ and $\beta = 10$. What is the probability that a rat survives no longer than 60 weeks?

Let the random variable X be the survival time (time to death). The required probability is

$$P(X \leq 60) = \frac{1}{\beta^5} \int_0^{60} \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(5)}\, dx.$$

The integral above can be solved through the use of the **incomplete gamma function**, which becomes the cumulative distribution function for the gamma distribution. This function is written as

$$F(x; \alpha) = \int_0^x \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)}\, dy.$$

If we let $y = x/\beta$, so $x = \beta y$, we have

$$P(X \leq 60) = \int_0^6 \frac{y^4 e^{-y}}{\Gamma(5)}\, dy,$$

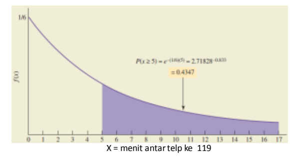
Contoh: Gamma

- Dari Tabel: $P(X \leq 60) = F(6;5) = 0.715.$

x	α = 1	2	3	4	5	6	7	8	9	10
1	0.6320	0.2640	0.0800	0.0190	0.0040	0.0010	0.0000	0.0000	0.0000	0.0000
2	0.8650	0.5940	0.3230	0.1430	0.0530	0.0170	0.0050	0.0010	0.0000	0.0000
3	0.9500	0.8010	0.5770	0.3530	0.1850	0.0840	0.0340	0.0120	0.0040	0.0010
4	0.9820	0.9080	0.7620	0.5670	0.3710	0.2150	0.1110	0.0510	0.0210	0.0080
5	0.9930	0.9600	0.8750	0.7350	0.5600	0.3840	0.2380	0.1330	0.0680	0.0320
6	0.9980	0.9830	0.9380	0.8490	0.7150	0.5540	0.3940	0.2560	0.1530	0.0840
7	0.9990	0.9930	0.9700	0.9180	0.8270	0.6990	0.5500	0.4010	0.2710	0.1700
8	1.0000	0.9970	0.9860	0.9580	0.9000	0.8090	0.6870	0.5470	0.4070	0.2830
9		0.9990	0.9940	0.9790	0.9450	0.8840	0.7930	0.6760	0.5440	0.4130
10		1.0000	0.9970	0.9900	0.9710	0.9330	0.8700	0.7800	0.6670	0.5420
11			0.9990	0.9950	0.9850	0.9620	0.9210	0.8570	0.7680	0.6590
12			1.0000	0.9980	0.9920	0.9800	0.9540	0.9110	0.8450	0.7580
13				0.9990	0.9960	0.9890	0.9740	0.9460	0.9000	0.8340
14				1.0000	0.9980	0.9940	0.9860	0.9680	0.9380	0.8910
15					0.9990	0.9970	0.9920	0.9820	0.9630	0.9300

Contoh: Eksponensial

- Jumlah telpon masuk pada nomor darurat 119 pada suatu kota diketahui berdistribusi Poisson dengan rata-rata 10 telpon per jam. Jika saat ini dilakukan pengamatan, berapakah peluang telpon masuk terjadi paling cepat 5 menit dari sekarang?
  - $\lambda = 10\ telpon\ per\ jam\ = 10/60\ telpon\ per\ menit$
  - $\beta = 1/\lambda = 6$ menit per telpon
  - $P(X \geq a) = e^{-\lambda a}$
  - $P(X \geq 5) = e^{-(\frac{1}{6})(5)} = 2{,}71828^{-0{,}833} = 0{,}4347$

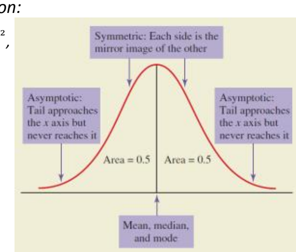


Distribusi Peluang Kontinyu

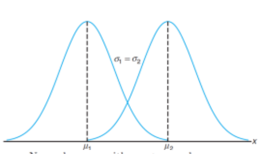
NORMAL

Distribusi Normal

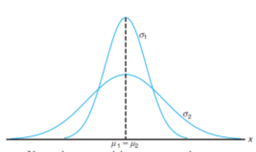
- *Gaussian distribution* (Karl Friedrich Gauss, 1777-1855)
- *Bell-shaped curve*
- *Probability density function:*
  - $n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$,
  - $-\infty < x < \infty$
  - $\pi = 3{,}14159 \ldots$
  - $e = 2{,}71828 \ldots$



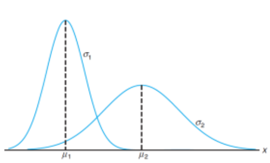
Distribusi Normal



Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$



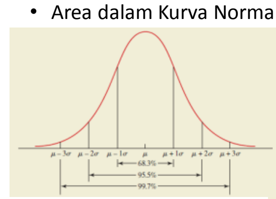
Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$

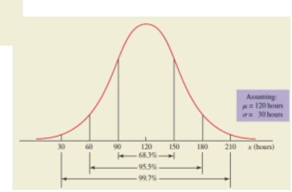


Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

Distribusi Normal

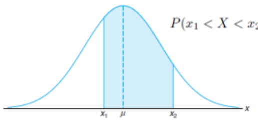
- Area dalam Kurva Normal



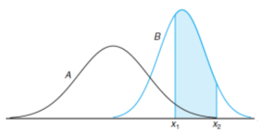


Distribusi Normal

- Area dalam Kurva Normal



$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x;\mu,\sigma)\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$



$P(x_1 < X < x_2)$ for different normal curves

Distribusi Normal

- Standard Distribusi Normal:
  - Kurva normal yang telah di-standarisasi dan menggambarkan nilai standar deviasi dari nilai rata-rata.
  - $Mean = 0, Variansi = 1. N(0,1).$
  - Z: normal random variabel dengan mean $= 0$, dan variansi $= 1$

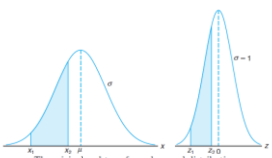
$$Z = \frac{X - \mu}{\sigma}$$

$$P(x_1 < X < x_2) = \frac{1}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx = \frac{1}{\sqrt{2\pi}}\int_{z_1}^{z_2} e^{-\frac{1}{2}z^2}\,dz$$

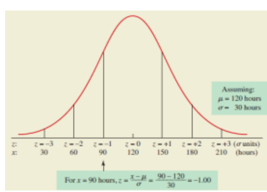
$$= \int_{z_1}^{z_2} n(z;0,1)\,dz = P(z_1 < Z < z_2),$$

$$z_1 = \frac{x_1 - \mu}{\sigma} \qquad z_2 = \frac{x_2 - \mu}{\sigma}$$

Distribusi Normal

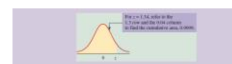


The original and transformed normal distributions



Distribusi Normal

- Menggunakan Tabel Distribusi Normal Standar



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

Distribusi Normal

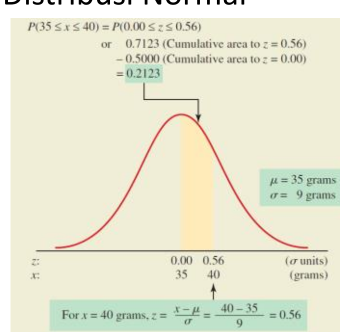
- Contoh Soal
  - Suatu perusahaan generator menghitung berat salah satu komponennya. Berat komponen tersebut berdistribusi normal dengan rata-rata 35 gram, dan standard deviasi 9 gram.
    1. Hitung probabilitas bahwa satu komponen yang diambil secara acak akan memiliki berat antara 35 dan 40 gram?
    2. Berapa peluang pengambilan acak satu komponen dengan berat paling ringan 50 gram?
  - JAWAB:
    1. **$P(35 \le x \le 40)$:**
       - $x = 40\ gram,$

         $z = \frac{x-\mu}{\sigma} = \frac{40-35}{9} = 0{,}56, P(Z \le 0{,}56) = 0{,}7123$
       - $x = 35\ gram,$

         $z = \frac{x-\mu}{\sigma} = \frac{35-35}{9} = 0, P(Z \le 0) = 0{,}5$
       - $P(35 \le x \le 40) = P(0 \le z \le 0{,}56) = 0{,}7123 - 0{,}5 = 0{,}2123$

Distribusi Normal

- Contoh Soal



Distribusi Normal

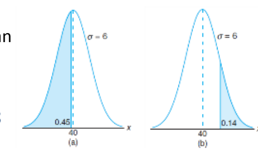
- Latihan Soal:
  – Nilai ujian fisika di sebuah kelas terdistribusi secara normal dengan rata-rata 60 dan standar deviasi 10. Berapa persen siswa yang memperoleh nilai antara 60 dan 70?

Distribusi Normal

- Menghitung nilai x

$$z = \frac{x-\mu}{\sigma}, \qquad maka\ x = z\sigma + \mu$$

- Contoh:
  – Diketahui suatu distribusi normal dengan $\mu = 40$ dan $\sigma = 6$. Carilah nilai x, yang memiliki:
    a. 45% area dari sisi kiri
    b. 14% area dari sisi kanan



Jawab:
a. $P(Z \leq z) = 0.45, z = -0{,}13$
$x = (6)(-0{,}13) + 40 = 39{,}22$

Distribusi Normal

- Latihan Soal:
  – Diketahui rata-rata hasil ujian adalah 74 dengan simpangan baku 7. Jika nilai-nilai peserta ujian berdistribusi normal dan 12% peserta nilai tertinggi mendapat nilai A, berapa batas nilai A yang terendah ?

Distribusi Normal

- Menyelesaikan permasalahan binomial dengan distribusi normal

$$Z = \frac{X - np}{\sqrt{npq}}$$

$n \to \infty$, is the standard normal distribution $n(z; 0, 1)$

Rangkuman

Distributions with Parameters	Possible Values of X	Density Function $f(x)$
Normal (μ, σ^2)	$-\infty < X < \infty$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$
Exponential (λ)	$0 < X$	$\lambda e^{-\lambda x}$
Gamma (α, β)	$0 < X$	$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

Note:
$P(x) = \int \boldsymbol{f(x)dx}$

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