PENELITIAN OPERASIONAL I

(TIN 4109)

Lecture 9

INTEGER LINEAR PROGRAMMING

Lecture 9 (Part 2)

- **Outline:**
  - Integer Linear Programming: Cutting Plane
- **References:**
  - Frederick Hillier and Gerald J. Lieberman. *Introduction to Operations Research*. 7th ed. The McGraw-Hill Companies, Inc, 2001.
  - Hamdy A. Taha. *Operations Research: An Introduction*. 8th Edition. Prentice-Hall, Inc, 2007.
  - Sriram, Sankaranarayanan. Computer Science, University of Colorado, Boulder. Http/www.coursera.org.
  - Winston, Wayne L. *Operations Research: Applications and Algorithms*. *3rd edition.* Wadsworth Inc.1994.

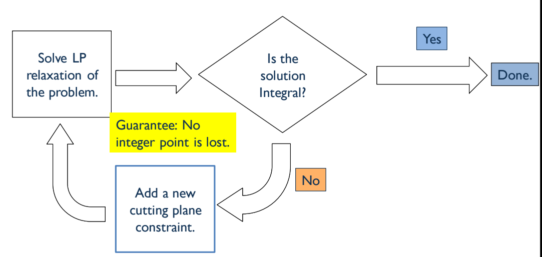
INTEGER PROGRAMMING

CUTTING PLANE METHOD

ILP: Cutting-Plane Algorithm

- Start at the continous optimum LP solution
- Add special constraint (called **cuts**) to renders an integer optimum extreme point
- The cuts do not eliminate any original feasible integer points
- The cuts must through at least one feasible or infeasible integer point
- Number of cuts is independent of the size of the problem

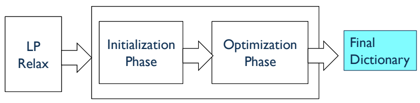
Overall method



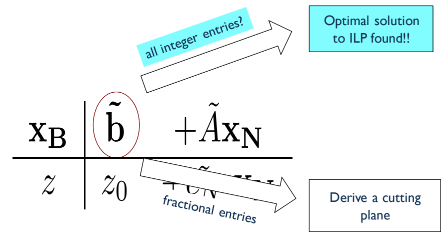
Overall Idea

$$\begin{array}{llll} \max & c^T x & & \\ \text{s.t.} & Ax + x_s & = & b \\ & x, x_s & \geq & 0 \\ & x, x_s & \in & \mathbb{Z} \end{array}$$

I. Solve the LP relaxation using Simplex algorithm.



Final Dictionary



ILP: Cutting-Plane Algorithm

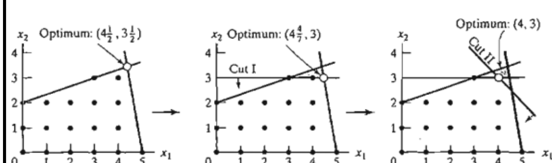
- Contoh permasalahan ILP:

Maximize $z = 7x_1 + 10x_2$

$$-x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_1, x_2 \geq 0 \text{ and integer}$$

ILP: Cutting-Plane Algorithm
-graphical approach-

- Solusi:



ILP: Cutting-Plane Algorithm
-algebra approach-

- Optimum LP tableau:

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$4\frac{1}{2}$

- Develops the cut
  - with assumption that all variables (including slacks) are integer

ILP: Cutting-Plane Algorithm
-algebra approach-

- Sources row: $z + \frac{63}{22}x_3 + \frac{31}{22}x_4 = 66\frac{1}{2}$ (z-equation)

  $x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = 3\frac{1}{2}$ (x_2-equation)

  $x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = 4\frac{1}{2}$ (x_1-equation)
- Factoring: x_2-*equation*
  - *factored as:* $x_2 + \left(0 + \frac{7}{22}\right)x_3 + \left(0 + \frac{1}{22}\right)x_4 = 3 + \frac{1}{2}$
- Factorial cut:

  $-\frac{7}{22}x_3 - \frac{1}{22}x_4 + \frac{1}{2} \leq 0$
- Cut's equation form:

  $-\frac{7}{22}x_3 - \frac{1}{22}x_4 + s_1 = -\frac{1}{2}, \quad s_1 \geq 0$ (Cut I)

ILP: Cutting-Plane Algorithm
-algebra approach-

- New tableau:

Basic	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	0	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$4\frac{1}{2}$
s_1	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$

- Solve with dual simplex
  - Result:

Basic	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	0	1	9	62
x_2	0	1	0	0	1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	

ILP: Cutting-Plane Algorithm
-algebra approach-

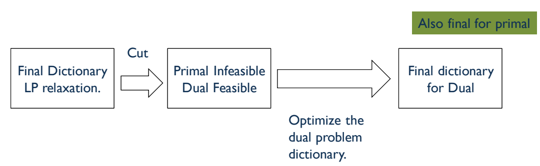
- Start to generate another cut
  - Until all variables are integer
- Final Result:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	0	3	7	58
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	−1	1	4
x_3	0	0	1	0	−4	1	1
x_4	0	0	0	1	6	−7	4

ILP: Cutting-Plane Algorithm
-algebra approach-

- **Steps to generate a cut:**
  - Choose a source row
  - Factoring the source row
  - Generate fractional cut
  - Form equation of the cut
- **Implement the cut**
  - Add constraint (equation) to the optimal tableu
  - Use dual simplex to solve the problem, if the tableau is optimal but infeasible
  - Start to generate another cut until all the variables are integer.

Cutting Plane:
Solving again after cut.



ILP: Cutting-Plane Algorithm
-important note-

- The fractional cut assumes that *all* the variables, *including slack and surplus*, are integers. This means that the cut deals with only the pure integer problem.
- There are 2 ways to remedy this situtiion:
  - Multiply the entire contraint by a proper constant to remove all the fraction
  - Use a special cut, called **mixed cut,** which allows only a subset of variables to assume integer values, with all the other variables remaining continous.

Integer Coefficients.

$$\begin{array}{llllll} \max & 2x_1 & +0.3x_2 & -0.1x_3 & & \\ \text{s.t.} & 0.1x_1 & -2x_2 & -x_3 & \leq & 0.25 \\ & 0.5x_1 & -2.6x_2 & +1.3x_3 & \leq & 0.15 \\ & x_1, & x_2, & x_3 & \geq & 0 \\ & x_1, & x_2, & x_4 & \in & \mathbb{Z} \end{array}$$

$$x_3$$

Scale constraints + Objective

$$\begin{array}{lllllll} \max & 2x_1 & +0.3x_2 & -0.1x_3 & & & \times 10 \\ \text{s.t.} & 0.1x_1 & -2x_2 & -x_3 & \leq & 0.25 & \times 20 \\ & 0.5x_1 & -2.6x_2 & +1.3x_3 & \leq & 0.15 & \times 100 \\ & x_1, & x_2, & x_3 & \geq & 0 & \\ & x_1, & x_2, & x_4 & \in & \mathbb{Z} & \end{array}$$

Conversion to Integer Coefficients.

$$
\begin{array}{llllll}
\max & 2x_1 & +0.3x_2 & -0.1x_3 & & \\
\text{s.t.} & 0.1x_1 & -2x_2 & -x_3 & \leq & 0.25 \\
 & 0.5x_1 & -2.6x_2 & +1.3x_3 & \leq & 0.15 \\
 & x_1, & x_2, & x_3 & \geq & 0 \\
 & x_1, & x_2, & x_4 & \in & \mathbb{Z}
\end{array}
$$

$$
\begin{array}{llllll}
\max & 20x_1 & +3x_2 & -x_3 & & \\
\text{s.t.} & 2x_1 & -40x_2 & -20x_3 & \leq & 5 \\
 & 50x_1 & -260x_2 & +130x_3 & \leq & 15 \\
 & x_1, & x_2, & x_3 & \geq & 0 \\
 & x_1, & x_2, & x_4 & \in & \mathbb{Z}
\end{array}
$$

Divide result by 10

ILP: Cutting-Plane Algorithm

-latihan soal-

Solve the following problems by the fractional cut, and compare the true optimum integer solution with the solution obtained by rounding the continuous optimum.

*(a) Maximize $z = 4x_1 + 6x_2 + 2x_3$

subject to

$$
\begin{aligned}
4x_1 - 4x_2 \quad &\leq 5 \\
-x_1 + 6x_2 \quad &\leq 5 \\
-x_1 + x_2 + x_3 &\leq 5 \\
x_1, x_2, x_3 \geq 0 &\text{ and integer}
\end{aligned}
$$

(b) Maximize $z = 3x_1 + x_2 + 3x_3$

subject to

$$
\begin{aligned}
-x_1 + 2x_2 + x_3 &\leq 4 \\
4x_2 - 3x_3 &\leq 2 \\
x_1 - 3x_2 + 2x_3 &\leq 3 \\
x_1, x_2, x_3 \geq 0 &\text{ and integer}
\end{aligned}
$$

Lecture 10 – Preparation

- **Materi:**
  - Metode Branch and Bound

SEE YOU