

# **PENELITIAN OPERASIONAL I**

(TIN 4109)

# Lecture 3

# LINEAR PROGRAMMING

# Lecture 3

- **Outline:**
  - Simplex Method
- **References:**
  - Frederick Hillier and Gerald J. Lieberman.  
*Introduction to Operations Research*. 7th ed. The McGraw-Hill Companies, Inc, 2001.
  - Hamdy A. Taha. *Operations Research: An Introduction*. 8th Edition. Prentice-Hall, Inc, 2007.

# Simplex Method:

- *Simplex*: Algoritma untuk menyelesaikan *LP*
- Dipublikasikan pertama kali oleh George B. Dantzig

G.B Dantzig: Maximization of a linear function  
of variables subject to linear inequalities, 1947

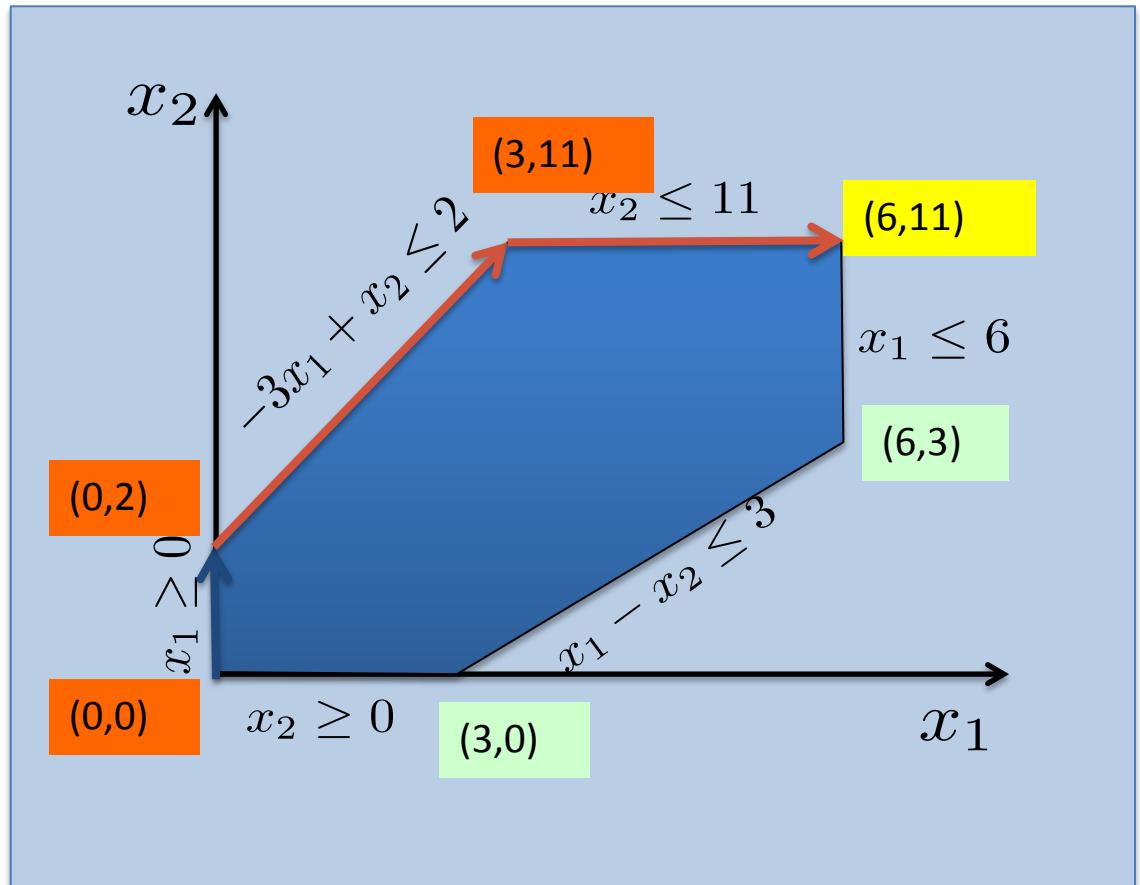


Photo credit:  
Stanford University

# Simplex Methods

## Visualizing

$$\begin{array}{lllll} \text{max.} & x_1 & +2x_2 & & \\ \text{s.t.} & -3x_1 & +x_2 & \leq & 2 \\ & & +x_2 & \leq & 11 \\ & x_1 & -x_2 & \leq & 3 \\ & x_1 & & \leq & 6 \\ & x_1, & x_2 & \geq & 0 \end{array}$$



# Linear Program

$$\begin{array}{llll} \text{maximize} & c_1x_1 + \dots + c_nx_n & & \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n & \leq & b_2 \\ & & \ddots & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n & \leq & b_m \\ & x_1, x_2, \dots, x_n & \geq & 0 \end{array}$$

# Linear Program in Matrix Form

$$\begin{array}{lll} \text{maximize} & c_1x_1 + \dots + c_nx_n \\ \text{subj.to.} & a_{11}x_1 + \dots + a_{1n}x_n & \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n & \leq b_2 \\ & \ddots & \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n & \leq b_m \\ & x_1, x_2, \dots, x_n & \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & \mathbf{c}^T \mathbf{x} \\ \text{subj.to.} & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

# Simplex Method:

## Standard LP Form

- **Properties:**
  - Semua konstrain adalah persamaan dengan nilai bukan negatif pada sisi kanan (*nonnegative*)
  - Semua variabel bernilai bukan negatif (*nonnegative*)
- **Langkah-langkah:**
  - Mengkonversi pertidaksamaan menjadi persamaan dengan nilai bukan negatif (*nonnegative*) pada sisi kanan
  - Mengkonversi unrestricted variabel menjadi bukan negatif (*nonnegative*) variabel

# Simplex Method: SLACK, SURPLUS, and Unrestricted variable

- **Definisi:**

- **SLACK variable ( $s_1$ ):**
  - variable yang menyatakan penggunaan jumlah kelebihan resources (*unused resources*) untuk menjadikan konstrain bertanda kurang dari ( $\leq$ ) menjadi persamaan (=).
  - Contoh:  $6x_1 + 4x_2 \leq 24$  menjadi  $6x_1 + 4x_2 + s_1 = 24; s_1 \geq 0$
- **SURPLUS / excess variable ( $S_1$ )**
  - variable yang menyatakan penyerapan persamaan sisi kiri untuk memenuhi batasan minimum resources sehingga menjadikan konstrain bertanda lebih dari ( $\geq$ ) menjadi persamaan (=).
  - Contoh:  $x_1 + x_2 \geq 800$  menjadi  $6x_1 + 4x_2 - S_1 = 800; S_1 \geq 0$
- **Unrestricted variable**
  - variable yang tidak memiliki batasan, dapat bernilai berapapun [(+); 0; atau (-)], dapat menggunakan slack dan surplus variable, secara matematis tidak jelas
  - maka unrestricted var.  $x_j$  perlu diubah menjadi  $x_j^+$  dan  $x_j^-$ ;  $x_j^+, x_j^- \geq 0$
  - Contoh:  $x_3$  unrestricted var. menjadi  $x_3 = x_3^+ - x_3^-$ ;  $x_3^+, x_3^- \geq 0$

# Simplex Method:

## Standard LP Form

- **Contoh:**

$$\text{Maximize } z = 2x_1 + 3x_2 + 5x_3$$

Subject to

$$\begin{aligned}x_1 + x_2 - x_3 &\geq -5 \\-6x_1 + 7x_2 - 9x_3 &\leq 4\end{aligned}$$

$$x_1 + x_2 + 4x_3 = 10$$

$$x_1, x_2 \geq 0$$

$x_3$  unrestricted



$$\text{Maximize } z = 2x_1 + 3x_2 + 5x_3^+ - 5x_3^-$$

Subject to

$$\begin{aligned}-x_1 - x_2 - x_3^+ + x_3^- + s_1 &= 5 \\-6x_1 + 7x_2 - 9x_3^+ + 9x_3^- + s_2 &= 4 \\x_1 + x_2 + 4x_3^+ - 4x_3^- &= 10 \\x_1, x_2, x_3^+, x_3^-, s_1, s_2 &\geq 0\end{aligned}$$



$$\text{Max. } z - 2x_1 - 3x_2 - 5x_3^+ + 5x_3^- + 0s_1 + 0s_2 = 0$$

Subject to

$$\begin{aligned}-x_1 - x_2 - x_3^+ + x_3^- + s_1 &= 5 \\-6x_1 + 7x_2 - 9x_3^+ + 9x_3^- + s_2 &= 4 \\x_1 + x_2 + 4x_3^+ - 4x_3^- &= 10 \\x_1, x_2, x_3^+, x_3^-, s_1, s_2 &\geq 0\end{aligned}$$

# Simplex Method:

## Transition from graphical to algebraic solution

### GRAPHICAL METHOD

Graph all constraints, including non negativity restriction

Solution space consists of infinity of **feasible points**

Identify **feasible corner points** of the solution space

Candidates for the optimum solution are given by a *finite* number of **corner points**

Use the objective function to determine the **optimum corner point** from among all the candidates

### ALGEBRAIC METHOD

Represent the solution space by  $m$  equations in the  $n$  variables and restrict all variables to nonnegativity values,  $m < n$

The system has infinity of **feasible** solutions

Determine the **feasible basic solutions** of the equations

Candidates for the optimum solution are given by a *finite* number of **basic feasible solutions**

Use the objective function to determine the **optimum basic feasible solution** from among all the candidates

# Simplex Method:

## Beberapa Definisi

- **Basic Variable**
  - Non zero-valued variable of basic solution
- **Non Basic Variable**
  - Zero-valued variable of basic solution
- **Basic Solutions**
  - solution obtained from standard LP with at most  $m$  non-zero
- **Basic Feasible Solutions**
  - a basic solution that is feasible
  - at most  $\binom{n}{m} = \frac{n!}{m!(n-m)!}; n > m$   
 $n = \text{number of variable}$   
 $m = \text{number of constraint}$
  - One of such solutions yields optimum if it exists
- **Adjacent basic feasible solution**
  - differs from the present basic feasible solution in exactly one basic variable

# Simplex Method:

## Beberapa Definisi

- **Simplex algorithm** moves from **basic feasible solution** to **basic feasible solution**; at each iteration it increases (does not decrease) the objective function value.
- **Pivot operation**
  - a sequence of elementary row operations that generates an adjacent basic feasible solution
  - chooses a variable to leave the basis, and another to leave the basis
- **Entering variable**
  - Non-basic variable with the most negative (most positive) coefficient for maximize (minimize) objective function in the z-row
  - Optimum is reached at the iteration where all z-row coefficients of the nonbasic variables are nonnegative (nonpositive) for maximize (minimize) function (**Optimality condition**)
- **Leaving Variable**
  - one of current basic variable that should be forced to zero level when entering level variable
  - chosen via a ratio test: the smallest (nonnegative) ratio (**Feasibility condition**)

# Simplex Method:

## Iterasi

- **Step 0:** Determine a starting basic feasible solution.
- **Step 1:** Select an entering variable using the optimality condition. Stop if there is no entering variable.
- **Step 2:** Select a leaving variable using the feasibility condition.
- **Step 3:** Determine the new basic solution by using the appropriate Gauss-Jordan computation. Go to step 1.

# Contoh Soal

Maximize  $z = 5x_1 + 4x_2$

Subject to:

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$-x_1 + x_2 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$



**Standard LP:**

$$\text{Maximize } z - 5x_1 - 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to:

$$6x_1 + 4x_2 + s_1 = 24$$

$$x_1 + 2x_2 + s_2 = 6$$

$$-x_1 + x_2 + s_3 = 1$$

$$x_2 + s_4 = 2$$

$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

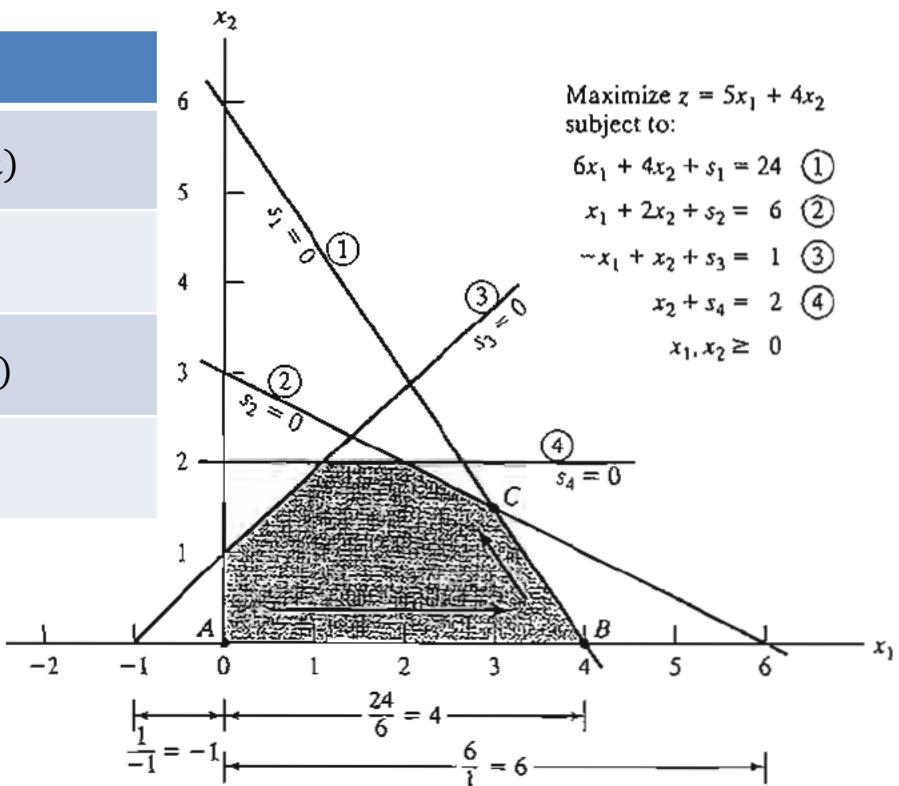


**TABEL SIMPLEX:**

Basic	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution	
$z$	1	-5	-4	0	0	0	0	0	$z$ -row
$s_1$	0	6	4	1	0	0	0	24	$s_1$ -row
$s_2$	0	1	2	0	1	0	0	6	$s_2$ -row
$s_3$	0	-1	1	0	0	1	0	1	$s_3$ -row
$s_4$	0	0	1	0	0	0	1	2	$s_4$ -row

Basic	$x_1$	Solution	Ratio
$s_1$	6	24	$\frac{24}{6} = 4$ (minimum)
$s_2$	1	6	$\frac{6}{1} = 6$
$s_3$	-1	1	$-\frac{1}{1} = -1$ (ignore)
$s_4$	0	2	$\frac{2}{0} = \infty$ (ignore)

Maximize  $z = 5x_1 + 4x_2$   
 subject to:  
 $6x_1 + 4x_2 + s_1 = 24 \quad (1)$   
 $x_1 + 2x_2 + s_2 = 6 \quad (2)$   
 $-x_1 + x_2 + s_3 = 1 \quad (3)$   
 $x_2 + s_4 = 2 \quad (4)$   
 $x_1, x_2 \geq 0$



Entering variable



Basic	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution	
$z$	1	-5	-4	0	0	0	0	0	z-row
$s_1$	0	6	4	1	0	0	0	24	$S_1$ -row
$s_2$	0	1	2	0	1	0	0	6	$S_2$ -row
$s_3$	0	-1	1	0	0	1	0	1	$S_3$ -row
$s_4$	0	0	1	0	0	0	1	2	$S_4$ -row

Leaving variable

**1. Pivot row**

- Replace the leaving variable in the *Basic* column with the entering variable.
- New pivot row = Current pivot row ÷ Pivot element

**2. All other rows, including z**

New Row = (Current row) - (Its pivot column coefficient) ×  
(New pivot row)

**STEPS:**

- Replace  $s_1$  in the *Basic* column with  $x_1$ :

$$\text{New } x_1\text{-row} = \text{Current } s_1\text{-row} \div 6$$

$$= \frac{1}{6}(0 \ 6 \ 4 \ 1 \ 0 \ 0 \ 0 \ 24)$$

$$= (0 \ 1 \ \frac{2}{3} \ \frac{1}{6} \ 0 \ 0 \ 0 \ 4)$$

2. New z-row = Current z-row -  $(-5) \times$  New  $x_1$ -row

$$= (1 \ -5 \ -4 \ 0 \ 0 \ 0 \ 0 \ 0) - (-5) \times (0 \ 1 \ \frac{2}{3} \ \frac{1}{6} \ 0 \ 0 \ 0 \ 4)$$

$$= (1 \ 0 \ -\frac{2}{3} \ \frac{5}{6} \ 0 \ 0 \ 0 \ 20)$$

$$3. \text{ New } s_2\text{-row} = \text{Current } s_2\text{-row} - (1) \times \text{New } x_1\text{-row}$$

$$= (0 \ 1 \ 2 \ 0 \ 1 \ 0 \ 0 \ 6) - (1) \times (0 \ 1 \ \frac{2}{3} \ \frac{1}{6} \ 0 \ 0 \ 0 \ 4)$$

$$\approx (0 \ 0 \ \frac{4}{3} \ -\frac{1}{6} \ 1 \ 0 \ 0 \ 2)$$

$$4. \text{ New } s_3\text{-row} = \text{Current } s_3\text{-row} - (-1) \times \text{New } x_1\text{-row}$$

$$= (0 \ -1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1) - (-1) \times (0 \ 1 \ \frac{2}{3} \ \frac{1}{6} \ 0 \ 0 \ 0 \ 4)$$

$$\approx (0 \ 0 \ \frac{5}{3} \ \frac{1}{6} \ 0 \ 1 \ 0 \ 5)$$

$$5. \text{ New } s_4\text{-row} = \text{Current } s_4\text{-row} - (0) \times \text{New } x_1\text{-row}$$

$$= (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 2) - (0)(0 \ 1 \ \frac{2}{3} \ \frac{1}{6} \ 0 \ 0 \ 0 \ 4)$$

$$\approx (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 2)$$

Basic	z	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution	
$z$	1	0	$-\frac{4}{6}$	$\frac{5}{6}$	0	0	0	20	New z = old z + (new $x_1$ value x its objective coefficient) = 0 + (4 x 5)
$x_1$	0	1	$\frac{4}{6}$	$\frac{1}{6}$	0	0	0	4	
$s_2$	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2	
$s_3$	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5	
$s_4$	0	0	1	0	0	0	1	2	

Basic	$x_2$	Solution	Ratio
$x_1$	$\frac{4}{6}$	4	$4 \div \frac{4}{6} = 6$
$s_2$	$\frac{4}{3}$	2	$2 \div \frac{4}{3} = \frac{3}{2}$ ( <i>minimum</i> )
$s_3$	$\frac{5}{3}$	5	$5 \div \frac{5}{3} = 3$
$s_4$	1	2	$\frac{2}{1} = 2$

Entering variable



Basic	z	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution
z	1	0	$-\frac{4}{6}$	$\frac{5}{6}$	0	0	0	20
$x_1$	0	1	$\frac{4}{6}$	$\frac{1}{6}$	0	0	0	4
$s_2$	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
$s_3$	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
$s_4$	0	0	1	0	0	0	1	2

← Leaving variable

1. New pivot  $x_2$ -row = Current  $s_2$ -row  $\div \frac{4}{3}$
2. New  $z$ -row = Current  $z$ -row  $- \left(-\frac{2}{3}\right) \times$  New  $x_2$ -row
3. New  $x_1$ -row = Current  $x_1$ -row  $- \left(\frac{2}{3}\right) \times$  New  $x_2$ -row
4. New  $s_3$ -row = Current  $s_3$ -row  $- \left(\frac{5}{3}\right) \times$  New  $x_2$ -row
5. New  $s_4$ -row = Current  $s_4$ -row  $- (1) \times$  New  $x_2$ -row

### HASIL OPTIMAL:

Basic	$z$	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	Solution
$z$	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
$x_1$	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
$x_2$	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
$s_3$	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
$s_4$	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

# Latihan Soal

Use the simplex algorithm to find the optimal solution to the following LP:

$$\min z = 4x_1 - x_2$$

$$\text{s.t. } 2x_1 + x_2 \leq 8$$

$$\text{s.t. } 2x_1 + x_2 \leq 5$$

$$x_1 - x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

# Latihan Soal

$$\begin{array}{lllll}\text{maximize} & 2x_1 + 3x_2 - 5x_3 \\ \text{s.t.} & x_1 - x_2 & \leq & 5 \\ & -x_1 + x_3 & \leq & 6 \\ & -2x_1 + x_3 & \leq & 2 \\ & -x_1 + x_2 & \leq & 4 \\ & x_1, x_2, x_3 & \geq & 0\end{array}$$

# Lecture 4 - Preparation

- **Read and Practice:**
  - Simplex: 2 Fase
    - Hamdy A. Taha. *Operations Research: An Introduction*. 8th Edition. Prentice-Hall, Inc, 2007. Chapter 3.

**SEE YOU**